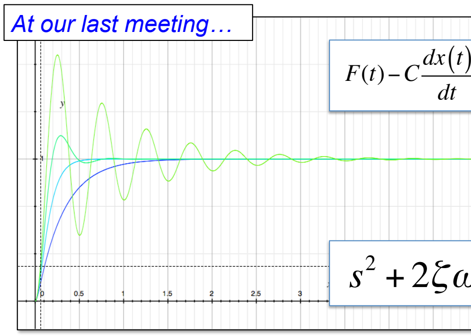
Lynbrook Robotics Team, FIRST 846

Damping & Resonance
2nd order differential equations

May 30 2012

Damping & Resonance
2nd order differential equation

At our last meeting…



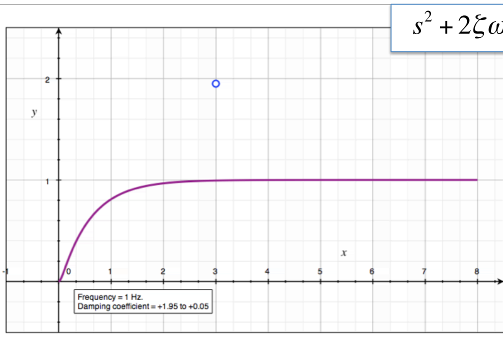
$$F(t)-C\frac{dx(t)}{dt}-k\,x(t)=m\frac{d^2x(t)}{dt^2}$$

$$s^2+2\zeta\omega_o s+\omega_o^2=0$$

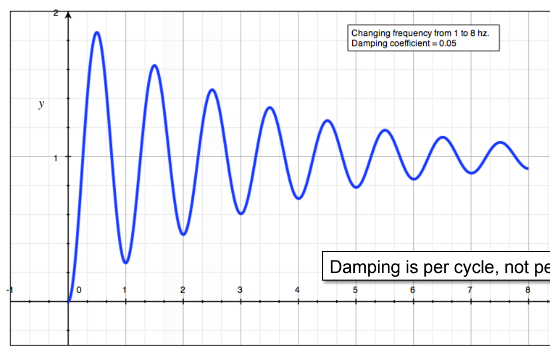
What do these mean?

Damping & Resonance

$$s^2+2\zeta\omega_o s+\omega_o^2=0$$



Damping & Resonance



Damping is per cycle, not per second

Simple Harmonic Oscillator

$$F_{mass}=ma=m\ddot{x}$$

$$F_{spring}=kx$$

$$F_{applied}-F_{spring}=m\,a$$

$$F(t)-k\,x(t)=m\frac{d^2x(t)}{dt^2}$$

Ref: http://en.wikipedia.org/wiki/Harmonic_oscillator

Simple Harmonic Oscillator

Look at system with no input (the "general" solution)

$$F(t)=0$$

$$0-k\,x(t)=m\frac{d^2x(t)}{dt^2}$$

$$-\frac{k}{m}x(t)=\frac{d^2x(t)}{dt^2}$$

Ref: http://en.wikipedia.org/wiki/Harmonic_oscillator

Simple Harmonic Oscillator

$$-\frac{k}{m}x(t)=\frac{d^2x(t)}{dt^2}$$

Solutions are of the form:

$$x(t)=\begin{cases}A\cos(\omega t)\\ B\sin(\omega t)\end{cases}\qquad \omega\equiv 2\pi f=\sqrt{\frac{k}{m}}$$

Simple Harmonic Oscillator

$$-\frac{k}{m}x(t)=\frac{d^2x(t)}{dt^2}$$

The math:

$$-\frac{k}{m}A\cos(\omega t)=\frac{d^2A\cos(\omega t)}{dt^2}$$

$$-\frac{k}{m}A\cos(\omega t)=\frac{-\omega\, A\, d\sin(\omega t)}{dt}$$

$$-\frac{k}{m}A\cos(\omega t)=-\omega^2\, A\cos(\omega t)$$

$$-\frac{k}{m}=-\omega^2$$

"Characteristic Equation"
(of linear homogeneous diff. equation)

$$\omega\equiv 2\pi f=\sqrt{\frac{k}{m}}$$

Ref: http://en.wikipedia.org/wiki/Characteristic_equation_%28calculus%29

Simple Harmonic Oscillator

$$-\frac{k}{m}x(t)=\frac{d^2x(t)}{dt^2}$$

Other solutions are of form:

$$x(t)=\begin{cases}C\,e^{j\omega t}\\ D\,e^{-j\omega t}\end{cases}$$

$$-\frac{k}{m}C\,e^{j\omega t}=\frac{d^2C\,e^{j\omega t}}{dt^2}$$

$$-\frac{k}{m}C\,e^{j\omega t}=j^2\omega^2C\,e^{j\omega t}$$

$$-\frac{k}{m}C\,e^{j\omega t}=(-1)\omega^2C\,e^{j\omega t}$$

Only 2 solutions in 2nd Order Diff.Eq.

- Euler's Formula

$$e^{j\theta}=\cos(\theta)+j\sin(\theta)$$

$$e^{-j\theta}=\cos(\theta)-j\sin(\theta)$$

$$\frac{e^{j\theta}+e^{-j\theta}}{2}=\cos(\theta)$$

$$\frac{e^{j\theta}-e^{-j\theta}}{2j}=\sin(\theta)$$

Ref: http://en.wikipedia.org/wiki/Euler%27s_formula

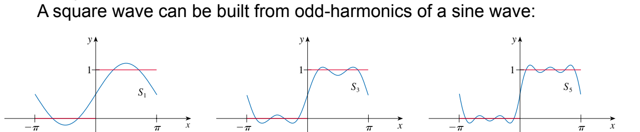
Building functions from sine waves

If F(t) ≠ 0

$$F(t)-k\,x(t)=m\frac{d^2x(t)}{dt^2}$$

…then build F(t) from sine / cosine functions.

Example:
A square wave can be built from odd-harmonics of a sine wave:



$\frac{1}{2}[1+\sin(x)]$ $\qquad$ $\frac{1}{2}[1+\sin(x)+\frac{1}{3}\sin(3x)]$ $\qquad$ $\frac{1}{2}[1+\sin(x)+\frac{1}{3}\sin(3x)+\frac{1}{5}\sin(5x)]$

Ref: www.stewartcalculus.com/data/.../upfiles/FourierSeriesEV.pdf

Harmonic Oscillator with Drag

$$F_{mass}=ma=m\ddot{x}$$

$$F_{spring}=kx$$

Add viscous friction (drag):
[Note: our systems more often have kinetic friction, not viscous friction]

$$F_{damper}=Cv=C\dot{x}$$

$$F_{applied}-F_{drag}-F_{spring}=ma$$

$$F(t)-C\frac{dx(t)}{dt}-k\,x(t)=m\frac{d^2x(t)}{dt^2}$$

Ref: http://en.wikipedia.org/wiki/Harmonic_oscillator

Harmonic Oscillator with Drag

$$F(t)-C\frac{dx(t)}{dt}-k\,x(t)=m\frac{d^2x(t)}{dt^2}$$

Find general solution; the solution without end conditions:

$$F(t)=0$$

$$0-C\frac{dx(t)}{dt}-k\,x(t)=m\frac{d^2x(t)}{dt^2}$$

Harmonic Oscillator with Drag

$$0-C\frac{dx(t)}{dt}-k\,x(t)=m\frac{d^2x(t)}{dt^2}$$

$$\frac{d^2x(t)}{dt^2}+\frac{C}{m}\frac{dx(t)}{dt}+\frac{k}{m}x(t)=0$$

Harmonic Oscillator with Drag

Find the characteristic equation:

$$\frac{d^2x(t)}{dt^2}+\frac{C}{m}\frac{dx(t)}{dt}+\frac{k}{m}x(t)=0$$

Choose the easy-to-handle solution: $x(t)=Ae^{st}$

$$s^2Ae^{st}+s\frac{C}{m}Ae^{st}+\frac{k}{m}Ae^{st}=0$$

$$s^2+s\frac{C}{m}+\frac{k}{m}=0$$

Harmonic Oscillator with Drag

Solve for s

$$s^2+s\frac{C}{m}+\frac{k}{m}=0$$

If C=0, we recognize this as the simple harmonic oscillator, where $\omega_o=\sqrt{\frac{k}{m}}$

$$s^2+\frac{C}{m}s+\omega_o^2=0$$

To solve for two roots s=r1, s=r2, I plan to use the quadratic formula

$$s^2+bs+c=0 \qquad with\ a=1;\ b=\frac{C}{m};\ c=\omega_o^2$$

Harmonic Oscillator with Drag

Solve for two roots s_2 and s_1:

$$s_1,s_2=\frac{-b\pm\sqrt{b^2-4\omega_o^2}}{2} \qquad a=1;\ b=\frac{C}{m};\ c=\omega_o^2$$

s_2 and s_1 are real if b ≥ $2\omega_o$

Introduce new variable ζ and <u>choose</u> b = $2\zeta\omega_o$

From our choice of b, we solve for ζ:

$$b=2\zeta\omega_o=\frac{C}{m}$$

so that: $\zeta=\frac{C}{2m\omega_o},\quad \omega_o=\sqrt{\frac{k}{m}}$

Harmonic Oscillator with Drag

<u>Choose</u> b = $2\zeta\omega_o$

$$s_1,s_2=\frac{-b\pm\sqrt{b^2-4\omega_o^2}}{2}$$

Then

$$\boxed{s^2+2\zeta\omega_o s+\omega_o^2=0}$$

$$s_1,s_2=-\zeta\omega_o\pm\sqrt{\zeta^2\omega_o^2-\omega_o^2}$$

s_2 and s_1 are complex if …. ?

Harmonic Oscillator with Drag

$$s_1, s_2 = \begin{cases} -\zeta\omega_o \pm \omega_o\sqrt{\zeta^2 - 1} & \zeta \geq 1 \\ -\zeta\omega_o \pm \boxed{j\omega_o\sqrt{1-\zeta^2}} & \zeta < 1 \end{cases}$$

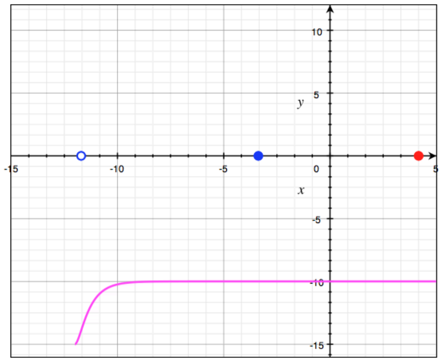
Imaginary → sine & cosine

If $\zeta < 1$, then there are solutions of the form: $e^{st} = e^{\sigma t} e^{\pm j\omega t}$:

$$x(t) = e^{-\zeta\omega_o t}\cos\left(\omega_o t\sqrt{1-\zeta^2}\right)$$

…*or* sin(…)

Harmonic Oscillator with Drag



moving roots.mov